Int. J. Solids Structures, 1969, Vol. 5, pp. 663 to 670. Pergamon Press. Printed in Great Britain

A DYNAMICAL EXPLANATION OF THE FALLING CAT PHENOMENON*

T. R. KANE and M. P. SCHER

Department of Applied Mechanics, Stanford University, California

IT is well known that falling cats usually land on their feet and, moreover, that they can manage to do so even if released from complete rest while upside-down. This phenomenon has given rise to questions of Dynamics as well as Physiology, and these have received attention in the literature of both fields [1–7]. In particular, numerous attempts have been made to discover a relatively simple mechanical system whose motion, when proceeding in accordance with the laws of Dynamics, possesses the salient features of the motion of the falling cat. The present paper constitutes such an attempt.

The phrase "salient features of the motion" requires elaboration, for its meaning is crucial to the determination of the extent to which a given theory can be regarded as successful. In order to be explicit on this point, we propose the following list of features (see Fig. 1):

(I) The torso of the cat bends, but does not twist.

(II) At the instant of release, the spine is bent forward. Subsequent to this instant, the spine is bent first to one side, then backward, then to the other side, and finally forward again, at which point the cat has turned over and the spine has the same shape as at the initial instant.

(III) The backward bend that occurs during the maneuver is far less pronounced than the initial and terminal forward bend.

Rademaker and Ter Braak [4] proposed a model capable of performing motions compatible with (I) and (II), but requiring equal backward and forward bending, and thus necessarily in conflict with (III). The present model accommodates all three requirements. Not surprisingly, this improvement can be obtained only at the expense of simplicity.

The system to be analyzed comprises two rigid bodies, A and B, which have one common point, O. To discuss the manner in which A and B move relative to each other, we introduce the following (see Fig. 2):

A_1, A_2, A_3	mutually perpendicular rays fixed in A and emanating from point O
K	a ray lying in the plane determined by A_1 and A_2
B_1	a ray fixed in body B
B_2	a ray perpendicular to B_1 and lying in the plane determined by B_1 and K (and not fixed in body B)
B_3	a ray perpendicular to both B_1 and B_2
N	a ray perpendicular to both A_1 and B_1
α	the angle between A_1 and K
β	the angle between B_1 and K
γ	the angle between A_1 and B_1
θ	the angle between A_3 and B_3

* This investigation was supported, in part, by NASA Research Grant NGR-05-020-209.

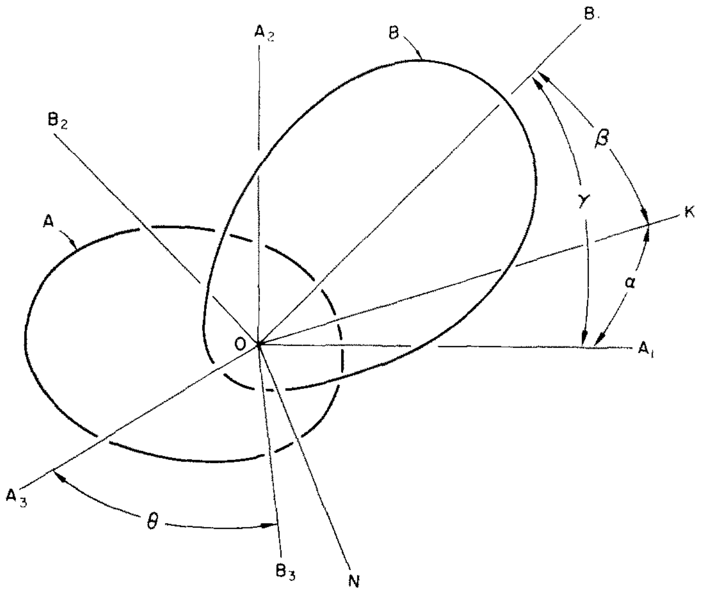

FIG. 2. Rays and angles.

$\mathbf{a}_i$	a unit vector parallel to A_i
$\mathbf{b}_i$	a unit vector parallel to B_i
$\mathbf{k}$	a unit vector parallel to K
$\mathbf{n}$	a unit vector parallel to N

A and B represent the front and rear halves of the cat, and A_1 and B_1 reflect the orientation of the spine. If A_2 is regarded as defining the ventral direction, forward or backward bending then takes place whenever B_1 lies in the A_1–A_2 plane, forward bending occurring when the angle between B_1 and A_2 is smaller than 90°, and backward bending when this angle is larger than 90°.

We now impose a requirement intended to reflect (I), that is, to eliminate twisting. To this end, we introduce a reference frame Q (see Fig. 3) in which N and the bisector of the angle between A_1 and B_1 are fixed, and note that ${}^Q\boldsymbol{\omega}^A$ and ${}^Q\boldsymbol{\omega}^B$, the angular velocities of A and B in Q, can be expressed as

$$ {}^Q\boldsymbol{\omega}^A = u\mathbf{a}_1 - (\dot{\gamma}/2)\mathbf{n} \tag{1} $$

and

$$ {}^Q\boldsymbol{\omega}^B = v\mathbf{b}_1 + (\dot{\gamma}/2)\mathbf{n} \tag{2} $$

where u and v are scalars that can be interpreted as "turning rates" of A and B in Q. Twisting is then prevented by setting

$$ v = u \tag{3} $$

Next, with (II) in mind, we stipulate that α and β remain constant. This means that B_1 is constrained to move on the surface of a right-circular cone of semi-vertex angle β,

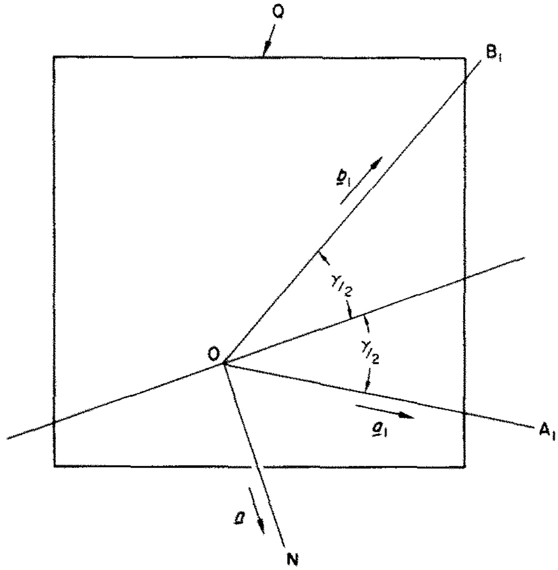

FIG. 3. Reference frame Q.

whose axis, K, is fixed in body A; and motions involving precisely the sequence of bending deformations of the spine described in (II) can now be generated in the following simple way: Taking $\beta > \alpha$, let B_1 revolve once in A about K or, equivalently, vary θ monotonically from zero to 2π radians.

Finally, we let the inertial orientation of the bisector of the angle between A_1 and B_1 remain unaltered throughout the motion. It follows that $\boldsymbol{\omega}^Q$, the inertial angular velocity of Q, can be expressed as

$$\boldsymbol{\omega}^Q = \dot{\psi} M(\mathbf{a}_1 + \mathbf{b}_1) \tag{4}$$

where $\dot{\psi}$ is the time-derivative of an angle ψ and M denotes the reciprocal of the magnitude of the vector $\mathbf{a}_1 + \mathbf{b}_1$. Furthermore, the term "overturning" can now be given a precise meaning: If ψ is chosen in such a way that $\psi = 0$ initially, then overturning has occurred when $\psi = \pm\pi$ if A and B each have the same orientation in Q at these two instants. A motion thus conforms to (II) in all respects if θ varies monotonically from zero to 2π, $\psi = 0$ when $\theta = 0$, and $\psi = \pm\pi$ when $\theta = 2\pi$.

As regards (III), all that can be said for the moment is that the backward bend associated with a motion of the kind just described is less pronounced than the forward bend, provided $\alpha > 0$, for the former is measured by $\beta - \alpha$ and the latter by $\beta + \alpha$.

As will be shown later, motions that satisfy all of the above requirements proceed in accordance with the laws of Dynamics if A_1 and B_1 are centroidal principal axes of inertia of A and B, respectively; the inertia ellipsoids of A and B are spheroids whose axes of symmetry are A_1 and B_1; the two bodies are identical; and ψ satisfies the differential equation

$$\frac{\mathrm{d}\psi}{\mathrm{d}\theta} = \frac{(J/I)S}{(T-1)[1-T+(J/I)(1+T)](1+T)^{\frac{1}{2}}} \tag{5}$$

where I and J denote the transverse and the axial moment of inertia of either body and S and T are given by

$$S = -\sqrt{2}(\cos\alpha\sin\beta+\sin\alpha\cos\beta\cos\theta)\sin\beta \tag{6}$$

$$T = \cos\alpha\cos\beta-\sin\alpha\sin\beta\cos\theta. \tag{7}$$

To study overturning by reference to equation (5), one may take $\psi(0) = 0$ and then determine $\psi(2\pi)$ by integrating the equation. This becomes particularly easy if $\alpha = 0$, for one can then obtain a solution in closed form; and if $\psi(2\pi)$ is set equal to π, this solution yields the relationship

$$2\sqrt{2}(J/I)(1+\cos\beta)^{\frac{1}{2}} = 1-\cos\beta+(J/I)(1+\cos\beta) \tag{8}$$

which is, essentially, a result obtained by Rademaker and Ter Braak [4]. It implies not only equal forward and backward bending, which conflicts with (III), but also a rather large amount of such bending, for with $J/I = 0{\cdot}25$, which is a realistic value,* equation (8) leads to a value of nearly 60° for β. It would appear, therefore, that zero is an unsatisfactory value for α.

Given J/I, one can proceed as follows to find pairs of values of α and β that permit overturning: Assign small values to α and β, and integrate equation (5) numerically in the interval $0 \leq \theta \leq 2\pi$. If $\psi(2\pi)$ is not equal to $\pm\pi$, increase β and integrate again, repeating this process until either a satisfactory value of β has been found or β has become so large as to be unacceptable on physical grounds; then increase α and begin a new search for β.

By performing such calculations with $J/I = 0{\cdot}2$ and $J/I = 0{\cdot}3$, one finds that overturning cannot occur unless there is some backbending, that is, unless $\beta < \alpha$. Figure 4 shows the result of our computations in the form of Forward Bend vs. Backbend plots for backbends up to 25°, which we regard as an upper limit from a physical point of view. These plots prove that the present theory accommodates (III). One may conclude, therefore, that this theory explains the phenomenon under consideration.

While the mechanical system which we have proposed is a rather simple one, the description of its motion, involving, in part, the solution of a nonlinear differential equation, is sufficiently complex to render visualization of the motion difficult. This difficulty can be overcome by drawing perspective pictures of bodies A and B for a number of instants during the motion. Representing A and B each as a right-circular cylinder, and indicating by means of two small crosses on each cylinder the points where one may imagine the legs of a cat to be attached to the torso, we have employed a computer-driven plotter† to create such pictures. For $J/I = 0{\cdot}25$, forward bending of 116°, and backbending of 25°, these values being estimates based on Fig. 1, the results appear as shown in Fig. 5. In Fig. 6, the same drawings have been superimposed on the photographs shown in Fig. 1.

We now return to the task deferred earlier, that of deriving equations (5)–(7) from a dynamical principle. To this end, we first define ω_i^A and ω_i^B as

$$\omega_i^A = \boldsymbol{\omega}^A \,.\, \mathbf{a}_i, \qquad \omega_i^B = \boldsymbol{\omega}^B \,.\, \mathbf{b}_i \tag{9}$$

* To determine J/I, measurements were performed, with the assistance of Dr. James Robinson of the NASA Ames Research Center, on fourteen segments of a dead cat.

† We gratefully acknowledge the aid of Mr. Mark Nelson and of the Computation Center of Stanford University in writing the plotting program.

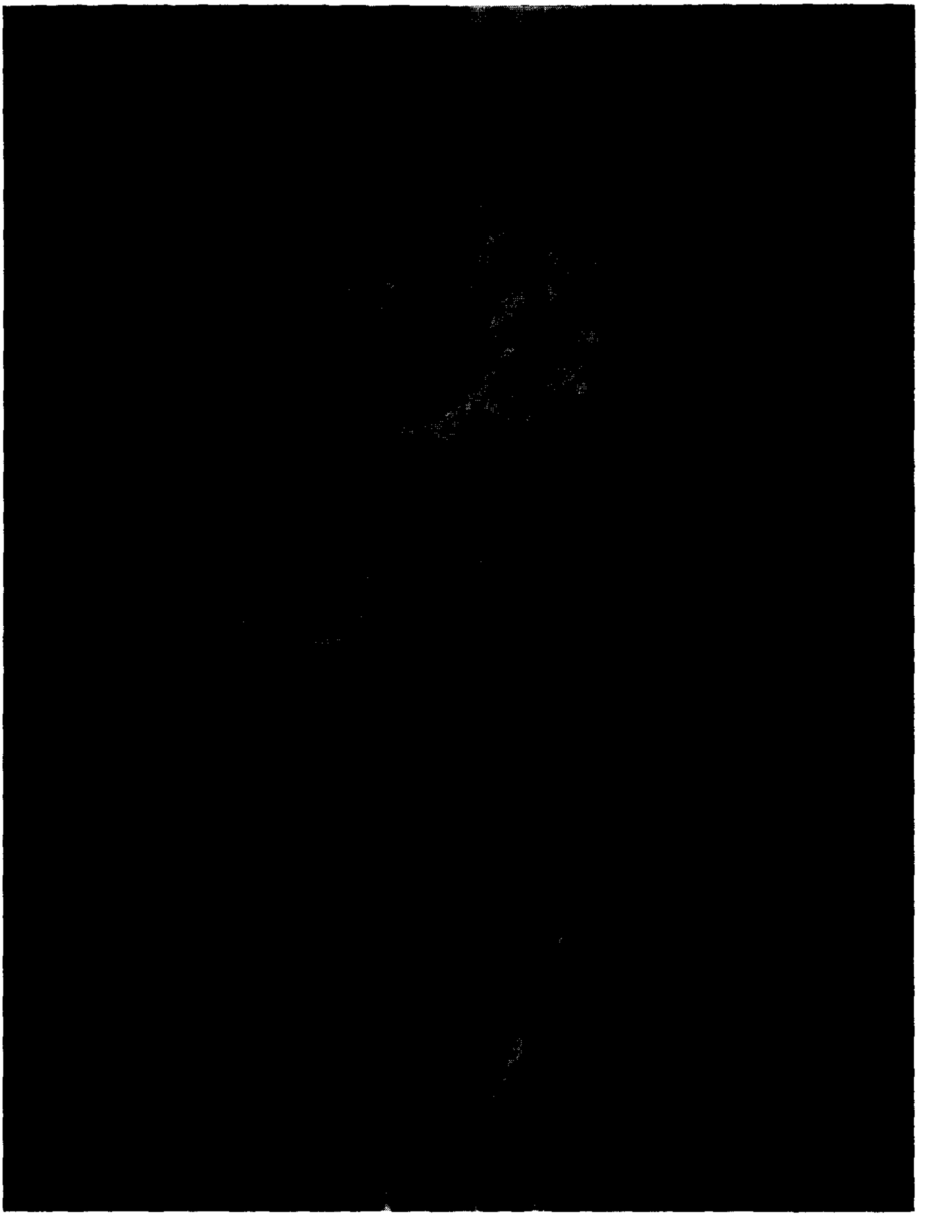

FIG. 1. Falling cat (Ralph Crane—*Life* magazine).

[*facing* p. 666

FIG. 6. Computer-drawn pictures overlaid on photographs of falling cat.

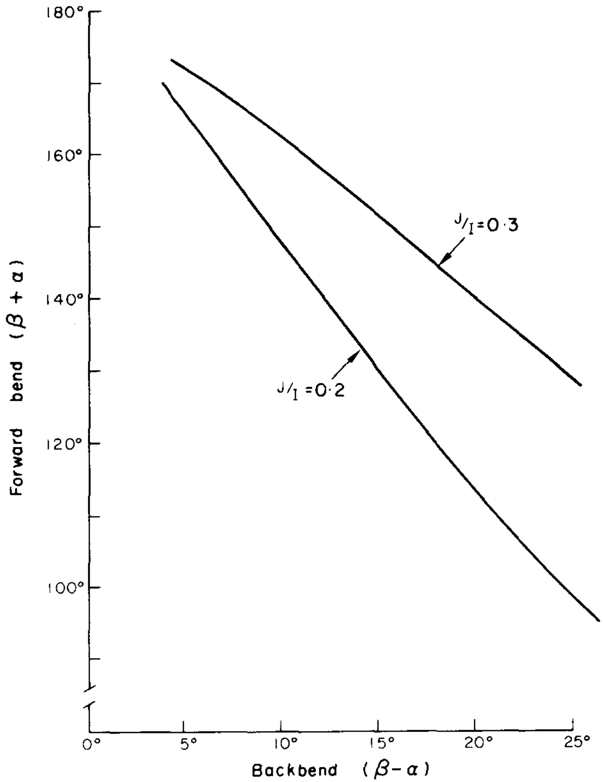

FIG. 4. Back bend and forward bend necessary for overturning.

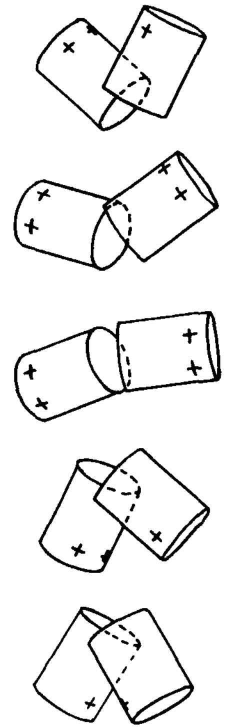

FIG. 5. Computer-drawn pictures of falling cat.

where $\boldsymbol{\omega}^A$ and $\boldsymbol{\omega}^B$ denote the inertial angular velocities of A and B. The fact that the angular momentum of the system relative to the mass center of the system must be equal to zero at all times (because it is initially equal to zero by hypothesis) can then be expressed as

$$J\omega_1^A\mathbf{a}_1 + I\omega_2^A\mathbf{a}_2 + I\omega_3^A\mathbf{a}_3 + J\omega_1^B\mathbf{b}_1 + I\omega_2^B\mathbf{b}_2 + I\omega_3^B\mathbf{b}_3 = 0 \tag{10}$$

and scalar multiplication of this equation with $\mathbf{a}_1 + \mathbf{b}_1$ yields

$$(J/I)(\omega_1^A + \omega_1^B)(1 + T_{11}) + \omega_2^A T_{21} + \omega_3^A T_{31} + \omega_2^B T_{12} + \omega_3^B T_{13} = 0 \tag{11}$$

where T_{ij} is defined as

$$T_{ij} = \mathbf{a}_i \,.\, \mathbf{b}_j. \tag{12}$$

Next, we seek to express ω_i^A and ω_i^B as functions of α, β, θ, $\dot{\theta}$, and $\dot{\psi}$.

Noting that*

$$\boldsymbol{\omega}^A \underset{(1,4)}{=} \boldsymbol{\omega}^Q + {}^Q\boldsymbol{\omega}^A = \dot{\psi}M(\mathbf{a}_1 + \mathbf{b}_1) + u\mathbf{a}_1 - (\dot{\gamma}/2)\mathbf{n} \tag{13}$$

and

$$\boldsymbol{\omega}^B \underset{(2,3,4)}{=} \boldsymbol{\omega}^Q + {}^Q\boldsymbol{\omega}^B = \dot{\psi}M(\mathbf{a}_1 + \mathbf{b}_1) + u\mathbf{b}_1 + (\dot{\gamma}/2)\mathbf{n} \tag{14}$$

we find by substitution into equations (9) and with the aid of (12) that

$$\omega_1^A = \dot{\psi}M(1 + T_{11}) + u \tag{15}$$

$$\omega_2^A = \dot{\psi}MT_{21} - (\dot{\gamma}/2)\mathbf{n} \,.\, \mathbf{a}_2 \tag{16}$$

$$\omega_3^A = \dot{\psi}MT_{31} - (\dot{\gamma}/2)\mathbf{n} \,.\, \mathbf{a}_3 \tag{17}$$

$$\omega_1^B = \dot{\psi}M(1 + T_{11}) + u \tag{18}$$

$$\omega_2^B = \dot{\psi}MT_{12} + (\dot{\gamma}/2)\mathbf{n} \,.\, \mathbf{b}_2 \tag{19}$$

$$\omega_3^B = \dot{\psi}MT_{13} + (\dot{\gamma}/2)\mathbf{n} \,.\, \mathbf{b}_3. \tag{20}$$

The unit vector $\mathbf{n}$ (see Fig. 2) can be expressed as

$$\mathbf{n} = \mathbf{a}_1 \times \mathbf{b}_1/\sin\gamma \tag{21}$$

and it follows that

$$\mathbf{n} \,.\, \mathbf{a}_2 = -\mathbf{a}_3 \,.\, \mathbf{b}_1/\sin\gamma = -T_{31}/\sin\gamma \tag{22}$$

$$\mathbf{n} \,.\, \mathbf{a}_3 = T_{21}/\sin\gamma \tag{23}$$

$$\mathbf{n} \,.\, \mathbf{b}_2 = T_{13}/\sin\gamma \tag{24}$$

$$\mathbf{n} \,.\, \mathbf{b}_3 = -T_{12}/\sin\gamma. \tag{25}$$

Of the nine quantities T_{ij}, only three are required in the sequel. By reference to equation (12) and Fig. 2, these can be expressed as

$$T_{11} = \cos\alpha\cos\beta - \sin\alpha\sin\beta\cos\theta \tag{26}$$

$$T_{12} = -\cos\alpha\cos\beta - \sin\alpha\cos\beta\cos\theta \tag{27}$$

$$T_{13} = \sin\alpha\sin\theta. \tag{28}$$

* Numbers beneath equal signs are intended to direct attention to corresponding equations.

An expression of $\dot{\gamma}$, suitable for later use, can now be constructed by noting that

$$\cos\gamma = \mathbf{a}_1 \cdot \mathbf{b}_1 = T_{11}$$

(12)

so that differentiation with respect to time yields

$$-\sin\gamma\dot{\gamma} = \dot{T}_{11} = \dot{\theta}T_{13}\sin\beta$$

(26,28)

or

$$\dot{\gamma} = -\dot{\theta}T_{13}\sin\beta/\sin\gamma \tag{29}$$

The quantity u that appears in equations (15) and (18) can be expressed as

$$u = \dot{\theta}T_{12}\sin\beta/(1-T_{11}^2). \tag{30}$$

This is shown as follows: ${}^A\boldsymbol{\omega}^B$, the angular velocity of B in a reference frame rigidly attached to A, is given both by

$${}^A\boldsymbol{\omega}^B = {}^Q\boldsymbol{\omega}^B - {}^Q\boldsymbol{\omega}^A = \dot{\gamma}\mathbf{n} + u(\mathbf{b}_1 - \mathbf{a}_1) \tag{31}$$

(1,2,3)

and by

$${}^A\boldsymbol{\omega}^B = {}^A\boldsymbol{\omega}^P + {}^P\boldsymbol{\omega}^B \tag{32}$$

where P designates a reference frame in which B_1 and K are fixed. Furthermore, the definition of θ is such that

$${}^A\boldsymbol{\omega}^P = \dot{\theta}\mathbf{k}.$$

And ${}^P\boldsymbol{\omega}^B$ must be parallel to $\mathbf{b}_1$, so that

$${}^P\boldsymbol{\omega}^B = s\mathbf{b}_1$$

where s is some scalar. Hence

$${}^A\boldsymbol{\omega}^B = \dot{\theta}\mathbf{k} + s\mathbf{b}_1. \tag{33}$$

(32)

Equating the right-hand members of equations (31) and (33), dot-multiplying the resulting equation with $\mathbf{b}_2$, and solving for u, one thus finds that

$$u = \frac{\dot{\gamma}\mathbf{n} \cdot \mathbf{b}_2 - \dot{\theta}\mathbf{k} \cdot \mathbf{b}_2}{\mathbf{a}_1 \cdot \mathbf{b}_2}$$

and use of equations (24), (29) and (12), together with the relationship $\mathbf{k} \cdot \mathbf{b}_2 = -\sin\beta$, then leads to equation (30).